Inhomogeneous Cosmological Models Containing Homogeneous Inner Hypersurface Geometry. Changes of the Bianchi Type.

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Abstract

There are investigated such cosmological models which instead of the usual spatial homogeneity property only fulfil the condition that in a certain synchronized system of reference all spacelike sections $t = \text{const.}$ are homogeneous manifolds. This allows time-dependent changes of the BIANCHI type. Discussing differential-geometrical theorems it is shown which of them are permitted. Besides the trivial case of changing into type I there exist some possible changes between other types. However, physical reasons like energy inequalities partially exclude them.

Es werden kosmologische Modelle betrachtet, die anstelle der üblichen räumlichen Homogenitätseigenschaft nur die Bedingung erfüllen, daß in einem gewissen synchronisierten Bezugssystem alle Raumschnitte $t = \text{const.}$ homogene Mannigfaltigkeiten sind. Neben dem trivialen Fall des Wechsels zu Typ I gibt es einige mögliche Änderungen zwischen anderen Typen. Physikalische Gründe wie Energieungleichungen schließen diese jedoch teilweise aus.

1 Introduction

Recently, besides the known BIANCHI models, there are investigated certain classes of inhomogeneous cosmological models. This is done to get a better representation of the really existent inhomogeneities, cf. e.g. BERGMANN (1981), CARMELI (1980), COLLINS (1981), SPERO (1978), SZEKERES (1975) and WAINWRIGHT (1981). We consider, similar as in COLLINS (1981), such inhomogeneous models V_4 which in a certain synchronized system of reference possess homogeneous sections $t = $ const., called $V_3(t)$ [1]. In the present paper we especially investigate which time-dependent changes of the BIANCHI type are possible. Thereby we impose, besides the twice continuous differentiability, a physically reasonable condition: the energy inequality, $T_{00} \geq |T_{\alpha\beta}|$, holds in each LORENTZ frame.

Under this point of view, we make some globally topological remarks: Under the physical condition the topology of the sections $V_3(t)$ is, according to LEE (1978), independent of t. Hence, the KANTOWSKI-SACHS models, with underlying topology $S^2 \times R$ or $S^2 \times S^1$ and the models of BIANCHI type IX with underlying topology S^3 or continuous images of it as $SO(3)$ may not change, because all other types are represented by the R^3-topology, factorized with reference to a discrete subgroup of the group of motions. But the remaining types can all be represented in R^3-topology itself; therefore we do not get any further global restrictions. All homogeneous models, the above mentioned KANTOWSKI-SACHS model being excluded, possess simply transitive groups (at least subgroups) of motion. Hence we do not specialize if we deal in the following only locally with simply-transitive groups of motions.[2]

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[1]This is analogous to the generalization of the concept of spherical symmetry in KRASINSKI (1980).

[2]We consequently do not consider here trivial changes of the BIANCHI type, e. g. from type III to type VIII by means of an intermediate on which a group of motions possessing

To begin with, we consider the easily tractable case of a change to type I: For each type M there one can find a manifold V_4 such that for each $t \leq 0$ the section $V_3(t)$ is flat and for each $t > 0$ it belongs to type M. Indeed, one has simply to use for all $V_3(t)$ such representatives of type M that their curvature vanishes as $t \to 0_+$.[3] Applying this fact twice it becomes obvious that by the help of a flat intermediate (of finite extension) or only by a single flat slice all BIANCHI types can be matched together. However, if one does not want to use such a flat intermediate the transitions of one BIANCHI type to another become a non-trivial problem. It is shown as well from the purely differential-geometrical as from the physical points of view (sections 3 and 4 resp.) which types can be matched together immediately without a flat intermediate. To this end we collect the following preliminaries.

2 Spaces possessing homogeneous slices

As one knows, in cosmology the homogeneity principle is expressed by the fact that to a space-time V_4 there exists a group of motions acting transitively on the spacelike hypersurfaces $V_3(t)$ of a slicing of V_4. Then the metric is given by

$$ds^2 = -dt^2 + g_{ab}(t)\omega^a\omega^b \tag{1}$$

transitive subgroups of both types acts.

[3]Choosing exponentially decreasing curvature one can obtain an arbitrarily high differentiable class for the metric, e. g.

$$ds^2 = -dt^2 + dr^2 + h^2 \cdot (d\psi^2 + \sin^2\psi d\phi^2),$$

where $h = r$ for $t \leq 0$ and

$$h = \exp(t^{-2}) \cdot \sinh^2\left(r \cdot \exp(-t^{-2})\right)$$

else. This is a C^∞-metric whose $V_3(t)$ belong to type I for $t \leq 0$ and to type V for $t > 0$. (In $t = 0$, of course, it cannot be an analytical one.) The limiting slice belongs by continuity causes necessarily to type I.

where the $g_{ab}(t)$ are positively definite and ω^a are the basic 1-forms corresponding to a certain BIANCHI type. If the ω^a are related to a holonomic basis x^i, using type-dependent functions $A_i^a(x^j)$, one can write:

$$\omega^a = A_i^a dx^i. \tag{2}$$

For the spaces considered here we have however: there is a synchronized system of reference such that the slices $t = $ const. are homogeneous spaces $V_3(t)$. In this system of reference the metric is given by

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \tag{3}$$

where $g_{ij}(x^i, t)$ are twice continuously differentiable and homogeneous for constant t. Hence it is a generalization of (1) and (2). This is a genuine generalization because there is only posed the condition $g_{0\alpha} = -\delta_{0\alpha}$ on the composition of the homogeneous slices $V_3(t)$ to a V_4. This means in each slice only the first fundamental form g_{ij} is homogeneous; but in the contrary to homogeneous models, the second fundamental form Γ_{0ab} need not have this property. Therefore, also the curvature scalar $^{(4)}R$ (and with it the distribution of matter) need not be constant within a $V_3(t)$. Now let t be fixed. In $V_3(t)$ one can find then coordinates $x_t^i(x^j, t)$ such that according to (1) and (2) the inner metric gets the form

$$g_{ab}(t)\omega_t^a \omega_t^b \qquad \text{where} \qquad \omega_t^a = A_i^a(x_t^i)dx_t^i. \tag{4}$$

Transforming this into the original coordinates x^i one obtains for the metric of the full V_4

$$g_{0\alpha} = -\delta_{0\alpha}, \qquad g_{ij} = g_{ab}(t)\, A_k^a(x_t^i)\, A_l^b(x_t^i)\, x_{t,i}^k\, x_{t,j}^l. \tag{5}$$

If the BIANCHI type changes with time one has to take such an ansatz for each interval of constant type separately; between them one has to secure a C^2-joining.[4]

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[4] An example of such an inhomogeneous model is (cf. ELLIS 1967)

$$ds^2 = -dt^2 + t^{-2/3}\left[t + C(x)\right]^2 dx^2 + t^{4/3}(dy^2 + dz^2).$$

3 Continuous changes of the BIANCHI type

Using the usual homogeneity property the same group of motions acts on each slice $V_3(t)$. Hence the BIANCHI type is independent of time by definition. But this fails to be the case for the spaces considered here. However, we can deduce the following: completing $\partial/\partial t$ to an anholonomic basis which is connected with KILLING vectors in each $V_3(t)$ one obtains for the structure constants[5] associated to the commutators of the basis:

$$C^0_{\alpha\beta} = 0\,; \quad C^i_{jk}(t) \quad \text{depends continuously on time.} \tag{7}$$

(Using only the usual canonical structure constants then of course no type is changeable continuously.)

To answer the question which BIANCHI types may change continuously we consider all sets of structure constants C^i_{jk} being antisymmetric in jk and fulfilling the JACOBI identity. Let C_S be such a set belonging to type S. Then we have according to (7): Type R is changeable continuously into type S, symbolically expressed by the validity of R $\to$ S, if and only if to each C_S there exists a sequence $C^{(n)}_R$ such that holds,

$$\lim_{n\to\infty} C^{(n)}_R = C_S\,. \tag{8}$$

The limit has to be understood componentwise.

It holds that R $\to$ S if and only if there are a C_S and a sequence $C^{(n)}_R$ such that (8) is fulfilled. The equivalence of both statements is shown by

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The slices $t = $ const. are flat, but only for constant C it belongs to BIANCHI type I.

[5] These structure constants are calculated as follows: Without loss of generality let $x^i_t(0,0,0,t) = 0$. At this point $\partial/\partial x^i_t$ and $\partial/\partial x^i$ are taken as initial values for KILLING vectors within $V_3(t)$. The structure constants obtained by these KILLING vectors are denoted by $\bar{C}^i_{jk}(t)$ and $C^i_{jk}(t)$ resp. where $\bar{C}^i_{jk}(t)$ are the canonical ones. Between them it holds at $(0,0,0,t)$:

$$x^i_{t,j}C^j_{kl} = \bar{C}^i_{mn}x^m_{t,k}x^n_{t,l}\,. \tag{6}$$

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means of simultaneous rotations of the basis. In practice one takes as C_S the canonical structure constants and investigates for which types R there can be found corresponding sequences $C_R^{(n)}$: the components of C_S are subjected to a perturbation not exceeding ϵ and their BIANCHI types are calculated. Finally one looks which types appear for all $\epsilon > 0$. Thereby one profits, e. g., by the statement that the dimension of the image space of the LIE algebra (0 for type I, ..., 3 for types VIII and IX) cannot increase during such changes.

One obtains the following diagram. The validity of R $\rightarrow$ S and S $\rightarrow$ T also imply the validity of R $\rightarrow$ T, hence the diagram must be continued transitively. The statement $VI_\infty \rightarrow IV$ expresses the facts that there exists a sequence

$$C_{VI_h}^{(n)} \rightarrow C_{IV}$$

and that in each such sequence the parameter h must necessarily tend to infinity.

Further $VI_h \rightarrow II$ holds for all h. Both statements hold analogously for type VII_h. In MACCALLUM (1971) it is shown a similar diagram but there it is only investigated which changes appear if some of the canonical structure constants are vanishing. So e. g., the different transitions from type VI_h to types II and IV are not contained in it.

Abb. 1

To complete the answer to the question posed above it must be added: to each transition shown in the diagram there one can indeed find a V_4 in which it is realized.[6] Actually, for the transition R $\rightarrow$ S the limiting slice belongs

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[6]Let $f(t)$ be a C^∞-function with $f(t) = 0$ for $t \leq 0$ and $f(t) > 0$ else. Then, e. g.,

$$ds^2 = -dt^2 + dx^2 + e^{2x}dy^2 + e^{2x}(dz + xf(t)dy)^2$$

is a C^∞-metric whose slices $V_3(t)$ belong to type V and IV for $t \leq 0$ and $t > 0$ resp. Presumably it is typical that in a neighborrbood of $t = 0$ the curvature becomes singular at spatial infinity.

necessarily to type S.

4 Physical conditions

To obtain physically reasonable space-times one has at least to secure the validity of an energy inequality, e.g. $T_{00} \geq |T_{\alpha\beta}|$ in each LORENTZ frame. Without this requirement the transition II $\to$ I is possible. We prove that the requirement $T_{00} \geq 0$ alone is sufficient to forbid this transition. To this end let V_4 be a manifold which in a certain synchronized system of reference possesses a flat slice $V_3(0)$ and for all $t > 0$ type II-slices $V_3(t)$. Using the notations of footnote 5 we have: $\bar{C}^1_{23} = -\bar{C}^1_{32} = 1$ are the only non-vanishing canonical structure constants of type II. With the exception $A^1_2 = -x^3$ we have $A^a_i = \delta^a_i$. By the help of (6) one can calculate the structure constants $C^i_{jk}(t)$. The flat-slice condition is equivalent to

$$\lim_{t \to 0} C^i_{jk}(t) = 0 \,. \tag{9}$$

First we consider the special case $x^i_t = a^i(t) \cdot x^i$ (no sum), i. e. extensions of the coordinate axes. It holds $x^i_{t,j} = \delta^i_j a^i$ (no sum). The only non-vanishing $C^i_{jk}(t)$ are $C^1_{23} = -C^1_{32} = a(t)$, where $a = a^2 \cdot a^3/a^1$. Then (9) reads $\lim_{t \to 0} a(t) = 0$. The coefficients $g_{ab}(t)$ have to be chosen such that the g_{ij}, according to (5), remain positive definite and twice continuously differentiable. Let $^{(3)}R$ be the scalar curvature within the slices. $^{(3)}R > 0$ appears only in type IX and in the KANTOWSKI-SACHS models. But changes from these types are just the cases already excluded by global considerations. Inserting g_{ij} into the EINSTEIN equation by means of GAUSS-CODAZZI theorem one obtains

$$\kappa T_{00} = R_{00} - \frac{1}{2} g_{00} \, ^{(4)}R = \frac{1}{2} \, ^{(3)}R + \frac{1}{4} g^{-1} \cdot H \,, \quad g = \det g_{ij} \,, \tag{10}$$

$^{(3)}R$ being the scalar curvature within the slices, hence

$$^{(3)}R \leq 0 \text{ and}$$

$$H = g_{11}[g_{22,0}g_{33,0} - (g_{23,0})^2] - 2 \cdot g_{12}[g_{33,0}g_{12,0} - g_{13,0}g_{23,0}] + \text{cyclic perm.} \quad (11)$$

In our case H is a quadratic polynomial in x^3 whose quadratic coefficient reads

$$-g_{11}[g_{11}g_{33} - (g_{13})^2] \cdot \left[\frac{\partial}{\partial t}a(t)\right]^2 .$$

Hence, for sufficiently large values x^3 and values t with $\partial/\partial t\, a(t) \neq 0$ we have $H < 0$, and therefore $T_{00} < 0$.

Secondly we hint at another special case, namely rotations of the coordinate axes against each other, e. g.

$$x_t^1 = x^1 \cos\omega + x^2 \sin\omega\,, \quad x_t^2 = x^2 \cos\omega - x^1 \sin\omega\,, \quad x_t^3 = x^3\,, \quad \omega = \omega(t)\,.$$

There one obtains in analogy negative T_{00}. Concerning the general case we have: loosely speaking, each diffeomorphism is a composite of such extensions and rotations. Hence, for each II $\rightarrow$ I-transition one would obtain points with negative T_{00}.

Hence, the energy condition is a genuine restriction to the possible transitions of the BIANCHI types. Concerning the other transitions we remark: equations (10) and (11) keep valid, and the remaining work is to examine the signs of the corresponding expressions H. Presumably one always obtains points with negative H, i.e., also under this weakened homogeneity presumptions the BIANCHI types of the space-like hypersurfaces at different distances from the singularity must coincide.

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$$\text{IX} \longrightarrow \text{VII}_0 \ \rule[0.5ex]{3cm}{1.5pt} \ \text{VII}_\infty$$

$$\text{VIII} \longrightarrow \text{VI}_0 \ \rule[0.5ex]{2.5cm}{1.5pt} \ \text{VI}_\infty$$

$$\text{II} \longleftarrow \text{IV}$$

$$\text{I} \longleftarrow \text{V}$$